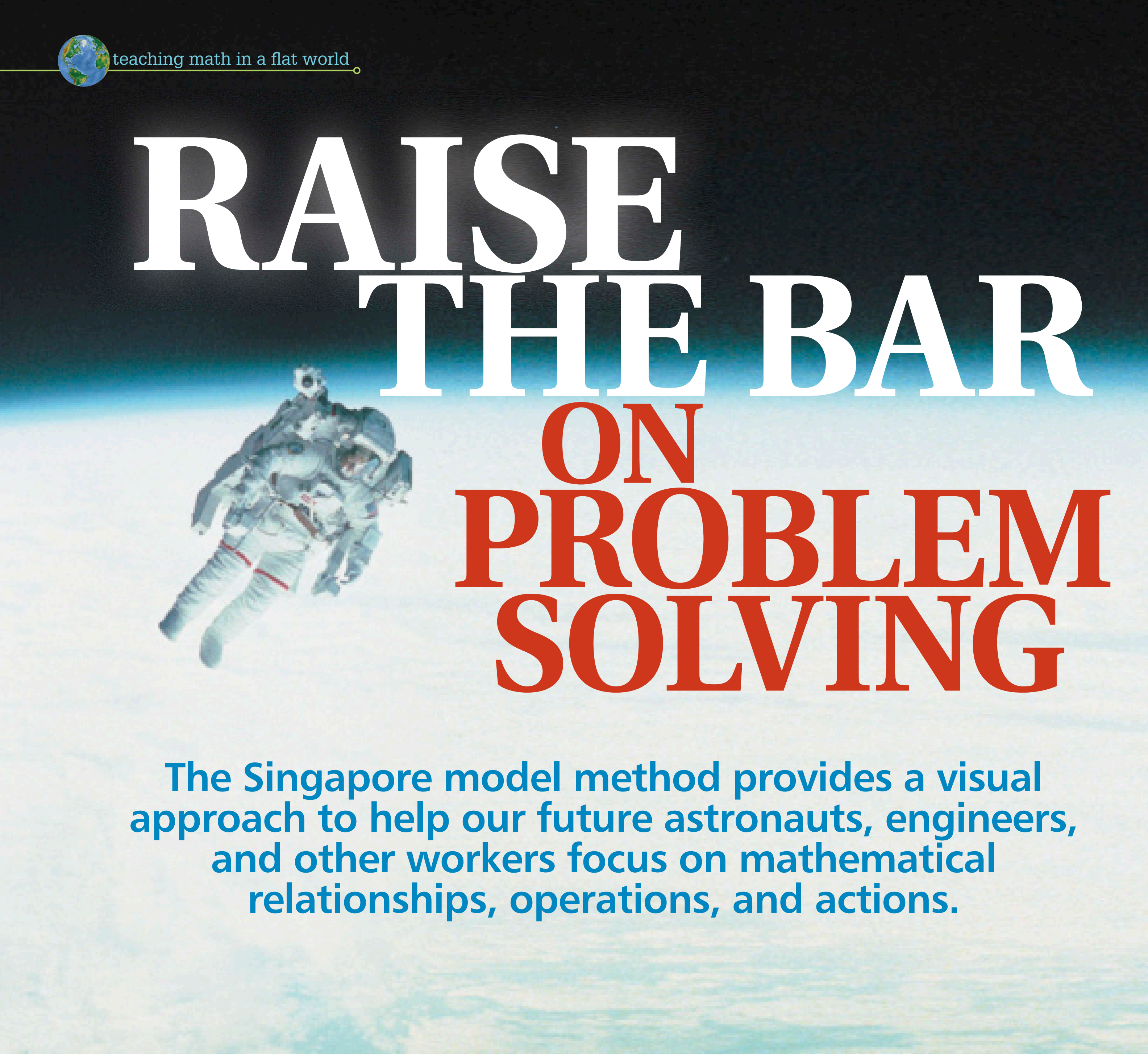


RAISE THE BAR ON PROBLEM SOLVING

The Singapore model method provides a visual approach to help our future astronauts, engineers, and other workers focus on mathematical relationships, operations, and actions.

By Lisa Englard

One of my favorite movie scenes is in *Apollo 13*, when NASA engineers, facing life-threatening carbon dioxide levels in the spacecraft, are charged with making a square filter fit into a round filter barrel using only items that the astronauts have on board. Using plastic bags intended for collecting moon rocks, cardboard from a flight-plan cover, spacesuit hoses, and duct tape, the engineers design a prototype and save the lives of the astronauts. Imagine the fate of James Lovell, Fred Haise, and John Swigert had those engineers been unable to problem solve creatively when faced with a new, unexplored situation.

As a math teacher, I take the responsibility of fostering creative problem solving very seriously. Who knows what future problems my students will be called upon to solve? But in my middle and high school classes, I encountered a troubling reality: With few exceptions, students could not make sense of word problems. I found this to be true over a wide range of abilities, from struggling students to those designated as gifted.

Research shows that young children naturally construct strategies for problem solving that model the actions and relationships presented in a problem. However, as they advance in mathematics, they begin looking for such superficial clues as which operation has just been taught, key words, and the numbers in the problem (Carpenter et al. 1999). Interestingly, in a 1981 diagnostic test, the Ministry of Education in Singapore found its country facing a similar challenge: Only 46 percent of students in grades 2–4 could solve word problems that were presented without such key words as *altogether* or *left* (Hong, Mei, and Lim 2009). Yet today, according to results from the Trends in International Mathematics and Science Study (TIMSS 2007), Singapore students continue to place among the best mathematical problem solvers in the world.

How did this small country improve its students' performance? It all began with an approach, developed by a team of educators, which advocated a concrete-to-pictorial-to-abstract strategy. The *model method*, as it is known in Singapore, has students draw a pictorial model to represent quantities and their relationships. The model puts the focus back on the relationships and actions presented in the problem, and helps students choose both the operations and sequence of steps that are needed to solve a problem.

To explore the possibilities of the model method, I worked with a third-grade class in our grades K–8 school during the final weeks of the school year. A pretest and a posttest with similar concepts were administered to all third-grade classes (see the posttest in the online **appendix**). Each test consisted of twenty word problems that began with one-step solutions, increasing in difficulty to multistep solutions. The nineteen-day intervention was a work in progress that evolved as I received feedback from students. I originally envisioned a focus on third grade only but later thought that gathering data from fourth- and fifth-grade classes as well would be interesting. Although there was no pretest or intervention program in those grades, I administered the posttest to all fourth- and fifth-grade students in order to gather data on varied approaches to problem solving.

The study results indicated that the test group was the only third-grade class to increase their problem-solving abilities and that the number of problems solved correctly by the test group after the intervention exceeded the averages for fourth- and fifth-grade students. Although the use of the model method was strongly encouraged in the test group, all strategies that produced a correct solution were accepted. Some students—in both the test group and the regular classes—shared their strategies, and some simply offered an answer. It was therefore sometimes difficult to determine solution strategies and whether they were conceptually sound. However, working with the model method clearly helped students in the test group tackle word problems.

The meaning of operations

How do we promote understanding of operations and when to use them in problem solving? Traditionally, students were taught to seek "key" words that point to the operation to use when attempting to solve the problem. This approach can mislead, as in this problem from the posttest:

> Skylar has 4 times as many books as Karen. If Skylar has 36 books, how many books does Karen have?

By following the number relationship presented in the problem, we see that Skylar has more books than Karen and that we need to divide thirty-six by four in order to find Karen's number of books. Students who focused on key words were incorrectly led to multiply by the word *times* (see **table 1**).

TABLE 1

Posttest results show that students can be misled by such key words in a problem as *times*.

Skylar has 4 times as many books as Karen. If Skylar has 36 books, how many books does Karen have?

Percentage of Students Who Multiplied Instead of Dividing	
Test group	0%
Grade 5	15%
Grade 4	26%
Grade 3	29%

FIGURE 1 The Singapore model method teaches students to represent quantities with bars of varying lengths and determine which operation to choose in one-step problems.

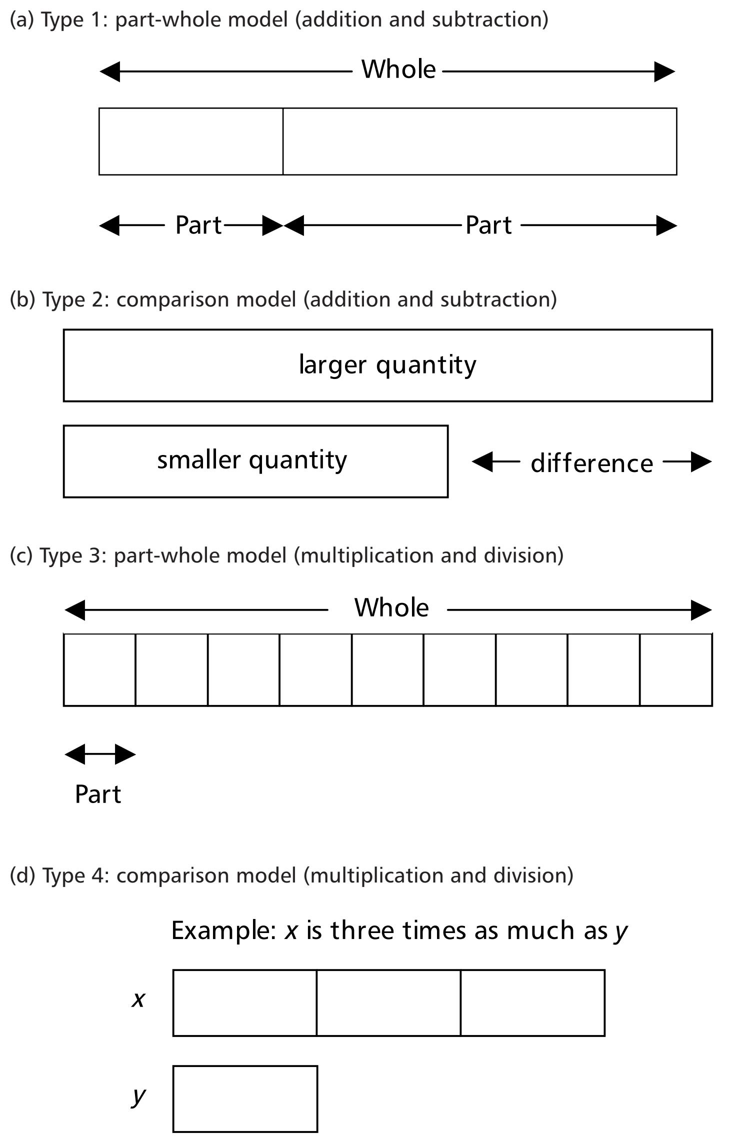


TABLE 2 Posttest results show that working with bar models helps students identify the correct operation to use.

Percentage of One-Step Problems with Wrong Operation	
Test group	3%
Grade 5	11%
Grade 4	20%
Grade 3	28%

In the test group, not one student chose multiplication, despite the key word *times*. Why did the test group have such success with this problem compared to the other classes? I believe it is because this group had benefited from the focus that the Singapore model method places on understanding the actions and the relationships in problems.

In using the method, students represent quantities with bars of varying lengths. Four basic models suffice to help students determine which operation to choose in one-step problems and greatly simplify the calculation of intermediate values in multistep problems:

1. To find the whole, add the parts.
   To find a missing part, subtract the known part from the whole (see **fig. 1a**).
2. To find the larger quantity, add the smaller quantity and the difference.
   To find the difference, subtract the smaller quantity from the larger quantity.
   To find the smaller quantity, subtract the difference from the larger quantity (see **fig. 1b**).
3. To find the whole, multiply the part by the number of equal partitions in the whole.
   To find the part, divide the whole by the number of equal partitions it contains (see **fig. 1c**).
4. To find *x*, multiply *y* by three.
   To find *y*, divide *x* by three (see **fig. 1d**).

Consider the two problem statements in **figure 2**. Notice their similarity and the use of the word *fewer* in both. Students who focus on identifying key words would likely be confused about whether to add or subtract. Students working with the bar models would draw addition-subtraction comparison models to represent the two problems.

The model makes it clear that the first problem calls for addition whereas the second requires subtraction. In both cases, we know the difference between the two numbers, which accounts for the similarity in the wording. The difference between the two problems lies in which quantity is known and which is missing. In the absence of a visual model that focuses on the relationships between the quantities, the words in the problem may confuse the solution strategy.

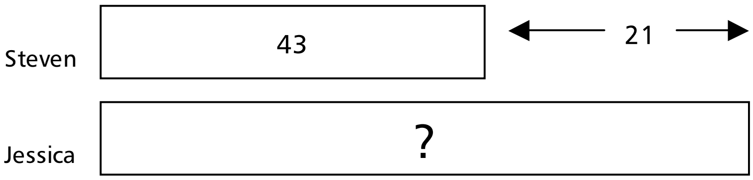
Returning to the posttest problem involving Skylar and Karen, a comparison model repre-

FIGURE 2

The wording of a problem may confuse students; a model can make it clear.

(a) The first problem calls for addition.

Steven has 21 fewer marbles than Jessica. If Steven has 43 marbles, how many does Jessica have?



(b) The second problem requires subtraction.

Mark has 30 fewer pieces of gum than Jordan. If Jordan has 200 pieces of gum, how much gum does Mark have?

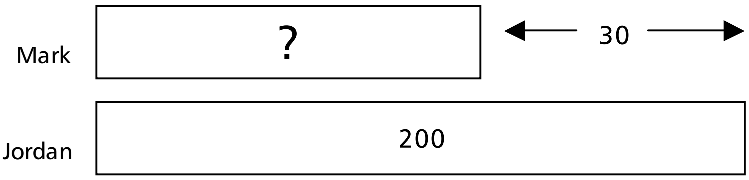
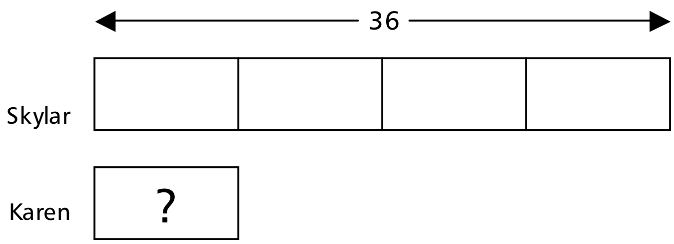


FIGURE 3

One feature of bar modeling is its emphasis on the relationships among operations. The model for the problem statement in **table 1** shows that multiplying the unknown quantity by four to get thirty-six is equivalent to dividing thirty-six by four.



senting the relationships in the problem makes it clear that we can find Karen's number of books by dividing thirty-six by four (see **fig. 3**). The word *times* in the problem statement pointed many students toward multiplication, yielding the result that Karen has more books than Skylar. This result contradicts the relationship in the problem statement. Additionally, one of the features of bar modeling is that it emphasizes the relationships among operations. By dividing thirty-six into four equal parts, the model shows that multiplying the unknown quantity by four to get thirty-six is equivalent to dividing thirty-six by four, thus focusing on the inverse nature of multiplication and division. Students who worked with the bar model strategy were more successful in identifying the correct operation to use when solving one-step problems with all operations (see **table 2**).

Multistep problems and algebraic thinking

Although the bar model method clarifies the choice of operation for one-step problems, its real beauty lies in the simplicity it lends to more complex, multistep problems. Students

FIGURE 4

The bar model method clarifies the choice of operation for one-step problems, but its real value is in the simplicity it lends to complex, multistep problems.

Jenny has 64 jellybeans. This is 8 times as many jellybeans as Carl has. How many more jellybeans does Jenny have than Carl?

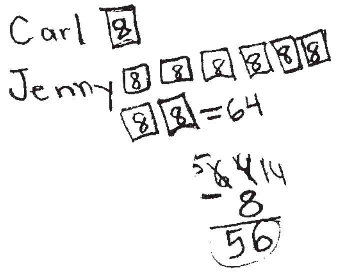


FIGURE 5

A bar model's visual nature makes it easy to see when to add or subtract, to keep track of each computation's outcome, and to fill in missing information.

James has 37 baseball cards. Paul has 23 more cards than James and 15 more cards than Ronald. How many cards do the three boys have altogether?

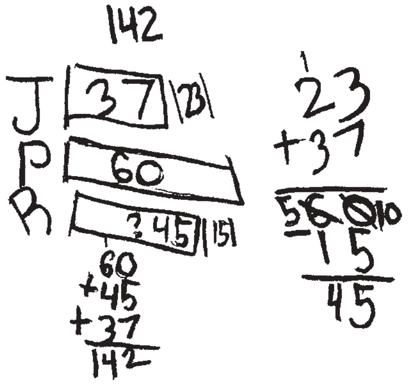
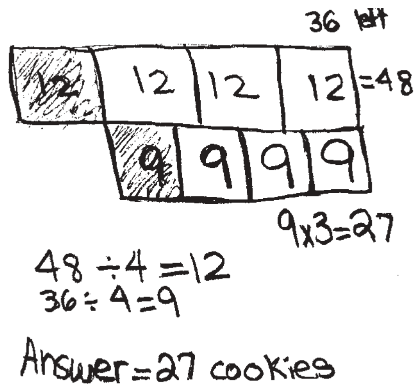


FIGURE 6

One question explored whether younger students could understand the concept of finding a fraction of a number without formally studying how to multiply fractions.

Mr. Davis made 48 cookies. He sold $\frac{1}{4}$ of them and gave $\frac{1}{4}$ of the remainder to his neighbor. How many cookies did he have left for himself?



who used the modeling approach were able to easily determine that Jenny's eight equal parts (see **fig. 4**), which total sixty-four, each represent eight jellybeans. Once the model was completed, they checked to see what question was asked and solved the problem with subtraction. Other students noted that Jenny has seven more parts than Carl, so she has 7 × 8, or 56, more jellybeans.

One student used the comparison model to compare three quantities (see **fig. 5**). The bars represent the number of cards for each boy; the numbers on the sides represent the differences in the quantities. From the problem statement, we know how many cards James has and some relationships between the boys' amounts. After setting up the model, this student used addition or subtraction as needed to fill in the missing pieces. The visual nature of the bar model makes it easy to see when to add or subtract and to keep track of what the outcome of each computation represents. Once the missing pieces of information were filled in, the student checked the problem statement to see what question needed to be answered. Finding the total for the three boys meant adding the numbers in their bars.

The problem in **figure 6** was meant to explore whether younger students could understand the concept of finding a fraction of a number without first formally studying how to multiply fractions. The results indicated that although all third, fourth, and fifth graders struggled with this problem, the test group was more successful after having explored the fraction model of the bar model method. The model simplified the problem by partitioning the quantities into fourths at each step and then dividing to find the parts.

Only 15 percent of the group of third-, fourth-, and fifth-grade students could solve the problem in **figure 7**, whereas over 40 percent of the test group solved it successfully. A student's work demonstrates the simplicity of the model. "Three times as many marbles as jump ropes" tells us to assign one unit bar to jump ropes and three bars of the same size to marbles. Four equal bars then sum to forty-four, so each bar represents eleven. Checking back with the question, there are 3 × 11, or 33, marbles.

An examination of incorrect solutions to the problem in **figure 7** revealed that most students disregarded the relationship presented in the problem and tried to perform an operation with the numbers presented. Many, misled by the word *times*, multiplied forty-four by three. Some divided forty-four by two because two items were discussed. Others divided forty-four by three and were confused by the remainder. Most students did not even attempt this problem. Among those with a correct solution, the majority presented their answer without showing any justification. Those who did show their work demonstrated an understanding of the three-to-one relationship and used either a guess-and-check strategy or displayed a running sum with two numbers, one of which was three times the other. A small number of students probably understood the relationship but answered the wrong question, offering "eleven" as the solution, which is the number of jump ropes. By encouraging students to label their quantities, the model method helps to avoid this particular error.

The student who completed the work in **figure 8** shows understanding of the concept that taking a fraction of a number means simultaneously partitioning the number into equal parts according to the denominator and collecting the number of groups given by the numerator. This student partitioned the whole group of fifty-four bandages into nine equal parts. Dividing fifty-four by nine yields six bandages for each part of the whole. To answer the question, she multiplied the remaining seven parts by six. Notice her multiplication facts above the bars.

FIGURE 7

Only 15 percent of third, fourth, and fifth graders could solve question 13; more than 40 percent of the test group was successful. Incorrect solutions revealed that most students disregarded the number relationships and instead tried to perform an operation with them: 44 × 3, 44 ÷ 2, or 44 ÷ 3.

Margo has three times as many marbles as jump ropes. If she has 44 marbles and jump ropes altogether, how many marbles does she have?

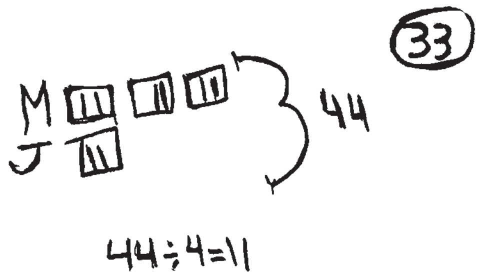


FIGURE 8

This student's work shows that she understands that fractions involve groups of equal parts. Notice her practice of multiplication facts.

A nurse has 54 bandages. $\frac{2}{9}$ of them are white and the rest are brown. How many of them are brown?

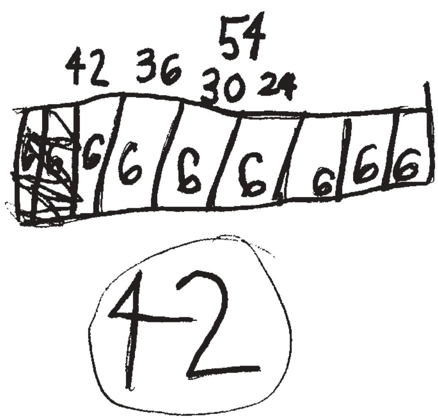


TABLE 3

Analysis of the posttest data shows that the test group was the only third-grade class to improve in problem solving, not only closing the gap but also outperforming their gifted peers.

Average Number of Problems Solved Correctly			
Students	**Pretest**	**Posttest**	**Change**
Gifted class	10	10	No change
Test group	7	11.5	+4.5
All other	5.75	4.75	–1

TABLE 4

The test group also outperformed older students on the posttest.

Number of Correct Problems	
Test group	11.5
Grade 5	11.4
Grade 4	9
Grade 3	7.8

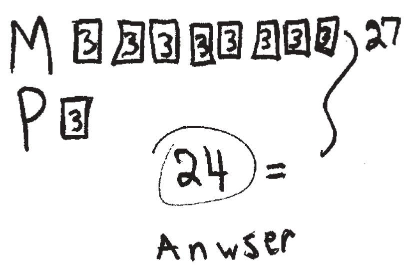
To solve the problem in **figure 9**, many older students multiplied 27 × 8 to get 216 and were unperturbed by their solution although it meant that Mike alone had more coins than the two boys had together. Others solved the problem but showed no work, suggesting that they probably used a mental guess-and-check strategy to find two numbers, one eight times the other, that add up to twenty-seven.

In our school, I have had the advantage of first watching students in grades K–8 develop through their elementary school years and then teaching them algebra I in seventh or eighth grade. My experience has been that many of these expert problem solvers encounter great struggles when numbers are replaced with unknown quantities in algebra. Having grown dependent on their facility with numbers, they are forced to understand the reasons behind the procedures in order to generalize the arithmetic to work with variables. Such students tend to resist writing equations to model situations and instead try to continue with their guess-and-check approaches. Even in third grade, according to the test group's classroom teacher, "typically high [level] math students struggled with the modeling concept. It seemed these students who can do math in their heads had a much more difficult time getting it down on paper."

FIGURE 9

Conceptually, this third grader provided an algebraic solution.

Mike and Paul have 27 coins altogether. If Mike has eight times as many coins as Paul, how many coins does Mike have?



The beauty of the modeling-with-bars strategy is that students essentially write an equation to model the situation but use a bar instead of a variable. For example, in the problem from **figure 9**, the equation would be $x + 8x = 27$, where x represents the number of coins that Paul has. To solve, we combine like terms (i.e., count the equal parts) to get $9x = 27$, then divide to get $x = 3$. Conceptually, the third grader whose work is shown provided an algebraic solution. Using the Singapore model method enhances the development of algebraic thinking and greatly simplifies the transition to working with variables.

Results

The test results reveal that the test group was the only third-grade class that improved in problem solving. Although these students lagged behind their gifted peers on the pretest, they were able to solve more problems successfully on the posttest than students in the gifted group (see **table 3**). On average, the test group outperformed the fourth- and fifth-grade classes (see **table 4**). On nineteen of the twenty problems on the posttest, the test group equaled or outperformed the aggregate group consisting of all third, fourth, and fifth graders (see **table 5**).

TABLE 5

The percentage of correct answers for each problem on the posttest shows that on average, the test group outperformed the third-, fourth-, and fifth-grade classes.

Percentage of Correct Answers																				
Problem no.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Test group	95	100	95	95	50	82	73	41	55	68	32	9	41	59	45	45	45	18	23	18
All 3, 4, 5	87	83	85	60	41	70	67	41	32	65	27	7	15	32	41	42	41	33	22	17

Helping teachers

Although the effect of the visual bar model method on student achievement is clear from the results of the posttest, an equally important consideration is the classroom teacher's reaction to the introduction of this new strategy. Research has shown that spending the bulk of mathematics instructional time to teach arithmetic skills does not prepare students to solve problems (Burns 2007). At the start of the intervention, when the students struggled with word problems, the teacher echoed this finding: "I feel like we should be doing it backwards. It's so sad that we spend so much time teaching them to compute, but it's worthless if they don't know how to apply it."

Bar modeling, when taught as a natural part of understanding operations (as it is in Singapore), provides an opportunity for teachers to infuse problem solving throughout their instruction. Rather than solving a few simple, one-step problems after pages of skills practice, as is common in many textbooks, the bar model method has students learning and understanding through problem solving and allows them to work on challenging multistep problems daily. The method builds throughout the grade levels to form a developmentally appropriate and vertically aligned concrete-to-pictorial-to-abstract strategy, helping students and teachers explore whole numbers, the meaning of operations, fractions, percents, ratios, and—finally—algebra.

My hope for the coming academic year is to introduce the bar model method to all students and teachers in grades K–8 at our school. Given the results of this nineteen-day intervention, imagine the growth that is possible when this method is presented and employed consistently throughout the elementary and middle school years. In the meantime, I am looking forward to seeing this third-grade group of mathematical thinkers in my algebra I honors class in four years. Who knows what future contribution they will make to solving the world's problems?

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Lisa Englard, lenglard@aventuracharter.org, is a math specialist for grades K–8 and teaches middle and high school mathematics at the Aventura City of Excellence School in Aventura, Florida. She is interested in problem solving and enjoys giving professional development workshops on learning through discovery and problem solving.

Posttest questions and an answer key accompany the online version of this article at **www.nctm.org/tcm/**.

Posttest

1. Connie has 33 marbles. Juan has 28 more marbles than Connie. How many marbles does Juan have?
2. There are 92 pumpkins and 38 cornstalks in a field. How many more pumpkins are there than cornstalks?
3. Grace lives 5 miles from the beach. Joey lives 8 times farther from the beach than Grace. How far does Joey live from the beach?
4. Skylar has 4 times as many books as Karen. If Skylar has 36 books, how many books does Karen have?
5. Jenny has 64 jellybeans. This is 8 times as many jellybeans as Carl has. How many more jellybeans does Jenny have than Carl?
6. James has 8 packs of baseball cards. Each pack has 7 cards. James gives 18 cards to his friend Steven. How many cards does James have left?
7. Alejandro paid $140 for his cable, phone, and Internet bills. The cable bill was $62. The phone bill was $59. How much was the Internet bill?
8. On Monday in the pet store there were 6 tanks that had 7 fish each and 3 tanks that had 12 fish each. On Tuesday, half the fish were sold. How many fish were left in the store on Wednesday?
9. For every paper boat that Harry makes, Lina can make 4 paper boats. If they make 35 boats altogether, how many paper boats does Harry make?
10. During summer vacation, Emma worked 82 hours, and Julie worked 33 hours. Don worked 18 more hours than Julie. How many more hours did Emma work than Don?
11. James has 37 baseball cards. Paul has 23 more cards than James and 15 more cards than Ronald. How many cards do the three boys have altogether?
12. Mr. Davis made 48 cookies. He sold 1/4 of them and gave 1/4 of the remainder to his neighbor. How many cookies did he have left for himself?
13. Margo has three times as many marbles as jump ropes. If she has 44 marbles and jump ropes altogether, how many marbles does she have?
14. Jim and Dan have $24 altogether. If Jim gives $6 to Dan, he will have three times as much money as Dan. How much money does Jim have?
15. Scott has a bag of candies. 2/5 of the candies are red. 1/5 of the candies are green. If there are 12 red candies, how many green candies are in the bag?
16. Claire saved 53 dollars. That was 29 dollars less than Henry saved. Jill saved 68 dollars. How many more dollars did Henry save than Jill?
17. Sue, Jane, and Alice share 27 paper clips equally. If Alice gives 6 paper clips to Sue, how many paper clips will Sue have?
18. Each box of 100 marbles has 37 blue marbles and 29 green marbles. The rest are yellow marbles. How many yellow marbles do 5 such boxes have?
19. A nurse has 54 bandages. 2/9 of them are white, and the rest are brown. How many are brown?
20. Mike and Paul have 27 coins altogether. If Mike has eight times as many coins as Paul, how many coins does Mike have?

Posttest Answer Key

1. 61 marbles
2. 54 more pumpkins
3. 40 miles
4. 9 books
5. 56 more jellybeans
6. 38 cards
7. $19
8. 39 fish
9. 7 boats
10. 31 hours
11. 142 cards
12. 27 cookies
13. 33 marbles
14. $24
15. 6 green candies
16. $14
17. 15 paper clips
18. 170 yellow marbles
19. 42 brown bandages
20. 24 coins